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The concept of sparse electron density as a tool in structure-factor phase determination*

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Abstract. From the fact that the space filled by electrons is only a fraction of the available space in non-centrosymmetrical crystal structures it can be anticipated that $\rho(\mathbf{r})\rho(-\mathbf{r}) \simeq 0$. This property has advantageous implications for phase determination. For instance, it may serve to resolve the twofold phase ambiguity that exists in single-wavelength anomalous-diffraction techniques.

Introduction. Structure-factor phase determination by direct methods is based on the fact that the electron density in crystals is positive everywhere and peaked at atomic positions. The last observation implies the existence of regions where electron density is sparse. So, for example, it can be expected that in non-centrosymmetric crystals $\rho(\mathbf{r})\rho(-\mathbf{r}) \simeq 0$. This assumption is corroborated by the fact that for those crystal structures the structure-factor phases are evenly distributed in the range $0-2\pi$. This leads to:

$$\int \rho(\mathbf{r})\rho(-\mathbf{r})d\mathbf{r} = \frac{1}{V} \sum_{\mathbf{h}} F_{\mathbf{h}}^2 \simeq F_0^2/V,$$

which is relatively small with respect to

$$\int \rho^2(\mathbf{r})d\mathbf{r} = \sum_{\mathbf{h}} |F_{\mathbf{h}}|^2/V. \quad (1)$$

We shall present here the first example of the use of the concept of sparse density.

Theory. Suppose that from the structure factors $F_{\mathbf{h}} = A_{\mathbf{h}} + iB_{\mathbf{h}}$ the $B_{\mathbf{h}}$ terms can be determined unambiguously. Such a situation is met within single-wavelength anomalous diffraction (SAD) provided that the anomalous scatterers are in a centrosymmetrical arrangement.

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Then the difference function $\rho(\mathbf{r}) - \rho(-\mathbf{r})$ can be calculated, since its Fourier coefficients read (see also Fan Hai-fu, Hao Quan & Woolfson, 1990)

$$\mathcal{F}_{\mathbf{h}}[\rho(\mathbf{r}) - \rho(-\mathbf{r})] = 2iB_{\mathbf{h}}. \quad (2)$$

In order to derive the $A_{\mathbf{h}}$ terms the sum function $\rho(\mathbf{r}) + \rho(-\mathbf{r})$ should be known:

$$2A_{\mathbf{h}} = \mathcal{F}_{\mathbf{h}}[\rho(\mathbf{r}) + \rho(-\mathbf{r})]. \quad (3)$$

Under the assumption that $\rho(\mathbf{r})\rho(-\mathbf{r}) = 0$ for all \mathbf{r} we can equate the squared sum and difference function because

$$[\rho(\mathbf{r}) + \rho(-\mathbf{r})]^2 = [\rho(\mathbf{r}) - \rho(-\mathbf{r})]^2 + 4\rho(\mathbf{r})\rho(-\mathbf{r}).$$

As a consequence, application of the convolution theorem leads to

$$\begin{aligned} \mathcal{F}_{\mathbf{h}}\{[\rho(\mathbf{r}) + \rho(-\mathbf{r})]^2\} &= \mathcal{F}_{\mathbf{h}}\{[\rho(\mathbf{r}) - \rho(-\mathbf{r})]^2\} \\ &= -4\sum_{\mathbf{k}} B_{\mathbf{k}}B_{\mathbf{h}-\mathbf{k}}. \end{aligned} \quad (4)$$

For an equal-atom structure we find from (2) and (4) the Sayre-type equation

$$A_{\mathbf{h}} = -2\frac{f_{\mathbf{h}}}{2f_{\mathbf{h}}V} \sum_{\mathbf{k}} B_{\mathbf{k}}B_{\mathbf{h}-\mathbf{k}}. \quad (5)$$

where $f_{\mathbf{h}}$ and $2f_{\mathbf{h}}$ are the scattering factors of the atoms and squared atoms respectively and V is the volume of the unit cell. Alternatively one could write

$$A_{\mathbf{h}} = \frac{-2f_{\mathbf{h}}}{\sum_{\mathbf{k}} f_{\mathbf{k}}f_{|\mathbf{h}-\mathbf{k}|}} \sum_{\mathbf{k}} B_{\mathbf{k}}B_{\mathbf{h}-\mathbf{k}}. \quad (6)$$

Table 1. Comparison of A_h values obtained by (7) with true values

| Compound | Z | Number of reflections | $\sum A - A_{\text{true}} / \sum A_{\text{true}} $ | Ref. |
|---|---|-----------------------|---|------|
| C ₂₀ H ₁₇ ClO ₇ | 1 | 3896 | 0.02 | (1) |
| C ₁₀ H ₁₃ N ₃ O ₄ S | 1 | 2401 | 0.04 | (2) |
| C ₆ H ₁₂ O ₄ S | 2 | 1852 | 0.03 | (3) |
| C ₂₃ H ₂₈ BrN ₂ O _{2.5} | 2 | 5076 | 0.01 | (4) |

References: (1) Gorst-Allman, Nolte & Steyn (1978); (2) Campaigne, Foltling, Huffman & Selby (1981); (3) Mathieson & Poppleton (1966); (4) Spek (1976).

Numerical tests. Test calculations were performed with artificial data obtained from the coordinates of known structures in $P1$, all with a centrosymmetric heavy-atom configuration. In this case

$$A_h = A_h^H - 2 \frac{f_h}{2f_h V} \sum_k B_k B_{h-k} \quad (7)$$

in which A_h^H refers to the heavy-atom contribution. For the scattering factors, temperature factors inclusive, the following analytical expressions were used:

$$f_h^B = \sum_{i=1}^5 a_i \exp \left[-\frac{1}{4} (b_i + B) h^2 \right]$$

and

$$2f_h^B = \sum_{i=1}^5 \sum_{j=1}^5 a_i a_j \left(\frac{4\pi}{b'_i + b'_j} \right)^{3/2} \times \exp \left[-\frac{1}{4} b'_i b'_j h^2 / (b'_i + b'_j) \right]$$

with $b'_i = b_i + B$. Coefficients a_i and b_i were taken from Cromer & Mann (1968) considering all light atoms as C. The results are given in Table 1, where the true A values are compared with those obtained by (7), in which the B_k were given their true values. The conclusion is that the condition $\rho(\mathbf{r})\rho(-\mathbf{r}) = 0$ is extremely well fulfilled in all four compounds.

An immediate practical application of (7) is the resolution of the twofold phase ambiguity in SAD. In the case of a centrosymmetric heavy-atom configuration, B_h can be derived from the Bijvoet inequalities (assuming the anomalous-scatterer substructure to be known), but A_h is known only in magnitude, not in sign. For the four known crystal structures A_h and B_h values were calculated from the Bijvoet differences, using error-free data and data contaminated with normally distributed errors; subsequently the sign of A_h was estimated by (7). Some of the results are given in Table 2. For comparison the results obtained by assuming (i) $\text{sign}(A_h) = \text{sign}(A_h^H)$ and (ii) $\phi_h = \phi_h^H$ (the heavy-atom method) are included in the table. It is apparent that the method works extremely well, at least

Table 2. Phase determination by single-wavelength anomalous diffraction.

Phase ambiguity resolved by (7) and by taking all A_h 's equal to the heavy-atom contributions respectively. In the last column the results from the heavy-atom method are given. Random errors in the intensities are indicated by standard deviations. For references see Table 1.

| Compound | $\sigma(I)^*$ | $\langle \phi - \phi_{\text{true}} \rangle$ using (7) ($^\circ$) | $\langle \phi - \phi_{\text{true}} \rangle$ [$\text{sign}(A_h) =$ $\text{sign}(A_h^H)$] ($^\circ$) | $\langle \phi - \phi_{\text{true}} \rangle$ [$\phi_h = \phi_h^H$] ($^\circ$) |
|---|---------------|--|--|--|
| C ₂₀ H ₁₇ ClO ₇ | 0.0 | 2.9 | 9.8 | 44.2 |
| | 3.0 | 16.3 | 21.9 | |
| | 4.4 | 21.8 | 27.3 | |
| C ₁₀ H ₁₃ N ₃ O ₄ S | 0.0 | 3.9 | 9.0 | 42.3 |
| | 0.0 | 3.6 | 10.2 | 38.0 |
| C ₂₃ H ₂₈ BrN ₂ O _{2.5} | 0.0 | 1.9 | 6.0 | 28.7 |

$$*\sigma(I) = 200[\langle |F_{\text{obs}}| - |F_{\text{calc}}| \rangle^2]^{1/2} / \langle |F_{\text{calc}}| \rangle.$$

with artificial data. Of course a number of practical tests with real data will have to be carried out to establish the quality of the method. Useful applications can be expected for large structures with relatively small heavy-atom contributions as the basic assumption, *i.e.* $\rho(\mathbf{r})\rho(-\mathbf{r}) \simeq 0$, does not depend on the size of the structure. We anticipate that it will be rewarding to derive the conditional probability density of A_h , given the B_h 's and $\rho(\mathbf{r})\rho(-\mathbf{r}) = 0$ for all \mathbf{r} 's.

Also direct-space procedures are to be considered, since the sparse density condition leads to

$$\rho(\mathbf{r}) + \rho(-\mathbf{r}) = |\rho(\mathbf{r}) - \rho(-\mathbf{r})|. \quad (8)$$

If the B_h 's are known, the right-hand side of (8) can be calculated and *via* image reconstruction, knowing the atomic shapes, be improved in order to lead to reliable A_h values.

Finally we arrive at the interesting conclusion that in favorable cases for non-centrosymmetrical structures only knowledge of the B_h values seems to be necessary in order to solve the phase problem. The presence of near-centrosymmetrical fragments with respect to the origin chosen will lead to discrepancies in (7). However, it should be noted that we are only interested in the sign of A_h and not in its precise magnitude.

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